What is a Derivative Geometrically?

In lecture we've slowly started to see that the derivative of a function can be understood as a matrix. In general let $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$ be of the form $\vec{f}(\vec{x}) = \langle f_1(\vec{x}), \cdots, f_m(\vec{x}) \rangle$ where $\vec{x} = \langle x_1, \cdots, x_n \rangle$. The full derivative is given by the $m \times n$ matrix

$$D\vec{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x^n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x^n} \end{pmatrix} = \begin{pmatrix} \nabla f_1 \\ \nabla f_2 \\ \vdots \\ \nabla f_m \end{pmatrix}$$

Why the heck is it a matrix??? Sure it might make notational sense in the very least but what's actually going on? The answer is somewhat subtle.

1. Let's start at the basics of some linear algebra to answer the above question. Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -3 & 5 \end{pmatrix}$$

It turns out that multiplication of a vector in $\mathbb{R}^2$ centered at the origin will give back a vector in $\mathbb{R}^2$ again. So we can think of A as being a map $A : \mathbb{R}^2 \to \mathbb{R}^2$. Let $\vec{v}_1 = (3, 1)^T$ and $\vec{v}_2 = (1, -1)^T$ (here the "T" notation means that we should really write the vectors standing upright instead of on their side. Vectors standing upright are called column vectors).

   (a) On a separate sheet of paper draw two copies of $\mathbb{R}^2$ side by side. On the first copy plot $\vec{v}_1$ and $\vec{v}_2$.

   (b) Compute the new vectors given by $\vec{w}_1 = A\vec{v}_1$ and $\vec{w}_2 = A\vec{v}_2$ and plot them on your second copy of $\mathbb{R}^2$. What is different about them compared to $\vec{v}_1$ and $\vec{v}_2$?

2. So what does this mean for a derivative? Let's work through an example. Let $\vec{r}(u,v) = \langle u\cos(\pi v/2), u\sin(\pi v/2), 9-u^2\rangle$ with $(u,v) \in [0,\infty) \times [0,4]$.

   (a) What shape is this parametric surface? Draw this shape on a separate piece of paper.

   (b) Compute $D\vec{r}$

   (c) Let $p = (2,4)$. What is $\vec{r}(p)$? Plot $\vec{r}(p)$ on the surface you drew in part (a). What is $D\vec{r}\Big|_p$?

   (d) You'll now notice that $D\vec{r}\Big|_p$ is just a matrix full of known numbers. Where does this matrix map from and where does it end?

(e) Now draw another picture of $\mathbb{R}^2$ (near your plot from part (a) if possible). Graph the point p and draw the vector $\vec{u} = \langle 3, 2 \rangle$ **starting** from the point p. Now $D\vec{r}\,\big|_p$ is the derivative of $\vec{r}$ at the point p. So $D\vec{r}\,\big|_p$ can multiply the vector $\vec{u}$. Compute $D\vec{r}\,\big|_p \vec{u}$. Where do you think the resulting vector should start from in $\mathbb{R}^3$?

(f) Compute a *parametric* equation of the tangent plane to the surface at $\vec{r}(p)$. Call it $\vec{K}(s,t)$ and rearrange it to look at $\vec{K}(s,t) - \vec{r}(p)$. What is $K(3,2) - \vec{r}(p)$? What does this mean about the vector $D\vec{r}\,\big|_p \vec{u}$ in relation to the surface?

Main Concept Question: If $\vec{f} : S_1 \to S_2$ where S_1 and S_2 are two surfaces in $\mathbb{R}^3$ then what does $D\vec{f}\,\big|_p$ map between for each $p \in S_1$?